n-point functions of $2d$ Yang-Mills theories on Riemann surfaces

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Abstract

Using the simple path integral method we calculate the n-point functions of field strength of Yang-Mills theories on arbitrary two-dimensional Riemann surfaces. In $U(1)$ case we show that the correlators consist of two parts , a free and an x-independent part. In the case of non-abelian semisimple compact gauge groups we find the non-gauge invariant correlators in Schwinger-Fock gauge and show that it is also divided to a free and an almost x-independent part. We also find the gauge-invariant Green functions and show that they correspond to a free field theory.

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Introduction

In recent years there have been many efforts to understand the two dimensional Yang-Mills theories. The partition function of these theories on Σ_g , a two-dimensional Riemann surface of genus g , has been calculated in the context of lattice gauge theory [1, 2] . On the other hand the string interpretation of $2d$ Yang-Mills theory was discussed in [3] and [4] by studying the $1/N$ expansion of the partition function for $SU(N)$ gauge group. It was shown that the coefficients of this expansion are determined by a sum over maps from a two-dimensional surface onto the two-dimensional target space.

Two-dimensinal Yang-Mills theories have also been studied by means of path integral method [5, 6]. In [7] and [8] the partition function and the expectation values of Wilson loops have been calculated and in [9 − 11] these quantities were calculated by using the abelianization technique.

In this paper we study the correlation functions of field strengths of $2d$ Yang-Mills theories by path integral method in a simple way. We drive all n-point functions on arbitrary surface. In the first part we consider the $U(1)$ gauge group and by calculating $Z[J]$, the partition function in the presence of an external source , we compute the gauge-invariant correlators on Σ_g. We show that the results consist of two parts , a part which comes from a free field theory and a second part which is independent of coordinates of fields. We also redrive the results by means of the expectation value of Wilson loops.

In the second part , we investigate the non-abelian gauge theories. Using the fermionic path integral representation of the trace of Wilson loops , we find the $Z[J]$ in Schwinger-Fock gauge and calculate the angle ordered $n-$point functions. We see that the non-gauge invariant correlators consist of a gauge-invariant part , and another part which is almost independent of the coordinates of the fields. We also extract the gauge-invariant part of the correlators and show that they correspond to a free theory. At the end we justify our results by using the Wilson loop correlators.

When this paper was nearly finished , we became aware of the preprint [13] in which the two and four-point functions of $U(N)$ gauge group have been derived by lengthy abelianization metod.

1 - Maxwell theory

Consider the partition function $Z[J]$ on Σ_g :

$$Z[J] = \int \mathcal{D}\xi e^{-\frac{1}{2\epsilon}\int \xi^2 d\mu + \int \xi J d\mu} \delta^p(\frac{1}{2\pi}\int \xi d\mu) \tag{1}$$

where the scalar field $\xi(x)$ is defined by $F_{\mu\nu} = \xi(x)\epsilon_{\mu\nu}, d\mu = \sqrt{g(x)}d^2x$ and

$$\delta^p(\frac{1}{2\pi}\int \xi d\mu) = \sum_n \delta(\frac{1}{2\pi}\int \xi d\mu - n) = \sum_n e^{in\int \xi d\mu}. \tag{2}$$

The insertion of $\delta^p(\frac{1}{2\pi}\int \xi d\mu)$ ensures that we are not integrating over arbitrary two-forms but over curvatures of connections. Performing the Gaussian integral (1) , we find

$$Z[J] = Z_1[J]Z_2[J], \tag{3}$$

where :

$$Z_1[J] = e^{\frac{\epsilon}{2}\int J^2 d\mu} \tag{4}$$

which is the partition function of a free field theory , and

$$Z_2[J] = \sum_n \exp[in\epsilon \int J d\mu - \frac{\epsilon}{2}n^2 A(\Sigma_g)] \tag{5}$$

in which $A(\Sigma_g)$ is the area of Σ_g. The gauge-invariant correlators of $\xi(x)$ is defined via :

$$< \xi(x_1)...\xi(x_n) >= \frac{1}{Z[0]}\frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}Z[J]|_{J=0} \tag{6}$$

Now Z_1 is the partition function of a free theory , so that its n-point functions are zero unless n is even. The $2n$-point functions factorize to the two point function

$$G(x,y) = \epsilon\delta(x-y). \tag{7}$$

So

$$< \xi(x_1)...\xi(x_{2n}) >_1 = \sum_p G(x_{i_1}, x_{i_2})...G(x_{i_{2n-1}}, x_{i_{2n}}), \tag{8}$$

where the summation is over all distinct pairings of the $2n$ indeces with $\frac{(2n)!}{(2!)^n n!}$ terms.

The correlators corresponding to Z_2 are also calculated to be

$$< \xi(x_1)...\xi(x_{2n}) >_2 = \frac{1}{Z[0]}\sum_m (im\epsilon)^{2n} e^{-\frac{\epsilon}{2}m^2 A(\Sigma_g)}, \tag{9}$$

and the odd-point functions are zero. The complete n-point function is simply obtained , using these two correlators. The odd-point functions are zero and the even-point functions are :

$$< \xi(x_1)...\xi(x_{2n}) >= \sum_{m=0}^{2n}\sum_c < \xi(x_1)...\xi(x_m) >_1 < \xi(x_{m+1})...\xi(x_{2n}) >_2, \tag{10}$$

where the inner summation is over all different ways of choosing m indeces from $2n$ indeces. As mentioned in the introduction the correlators consist of a free and an x-independent parts.

Now it is useful to reproduce the above results from another method , that is using the expectation value $< e^{i\alpha\oint_\gamma A} >$ where γ is a homologically trivial loop on Σ_g :

$$< e^{i\alpha\oint_\gamma A} >= \frac{1}{Z[0]}\int \mathcal{D}\xi e^{-\frac{1}{2\epsilon}\int \xi^2 d\mu + i\alpha\int_D \xi d\mu}\delta^p(\frac{1}{2\pi}\int \xi d\mu)$$

$$= \frac{1}{Z[0]}\sum_n exp\{-\frac{\epsilon}{2}[n^2 A(\Sigma_g) + \alpha^2 A(D) + 2\alpha n A(D)]\}$$

$$= 1 + \alpha^2 \frac{\epsilon/2}{Z[0]}\sum_n (\epsilon n^2 A^2(D) - A(D))e^{-\frac{\epsilon}{2}n^2 A(\Sigma_g)} + o(\alpha^2), \tag{11}$$

where $A(D)$ is the area of disk D , the boundary of which is γ. Expanding the left hand side of eq.(11) in terms of α , gives

$$< e^{i\alpha\int_D A} >= 1 + i\alpha\int_D < \xi(x) > d\mu - \frac{\alpha^2}{2}\int_D < \xi(x)\xi(y) > d\mu(x)d\mu(y) + o(\alpha^2). \tag{12}$$

By comparing the two side of eq.(11) , it is seen that $< \xi(x) >= 0$. In order to find the two point function we use the following ansatz

$$< \xi(x)\xi(y) >= M\delta(x-y) + N. \tag{13}$$

In fact , as the theory is topological , it is reasonable that the two point function consists of an x-independent term and a term which just sees if the two point are equal or not. This term should be a delta term , because of the Gaussian nature of the integrand in the partition function. Using this ansatz , it is readily seen that

$$< \xi(x)\xi(y) >= \epsilon\delta(x-y) - \frac{\epsilon^2}{Z[0]}\sum_m m^2 e^{-\frac{\epsilon}{2}m^2 A(\Sigma_g)}, \tag{14}$$

which is the result obtained previously. Other n-point functions can also be obtained in this way.

2 - Yang-Mills theory

In this section we are going to calculate $Z[J]$ for a non-abelian semisimple compact gauge group G. To begin , we consider the wavefunction $\psi_D[J]$ on the disk D. If γ is the boundary of D , $\gamma = \partial D$, we choose the boundary condition to be $\mathrm{Pexp}\oint_\gamma A = g_1 \in G$ (modulo conjugation) , and therefore $\psi_D[J]$ is defined as :

$$\psi_D[J] = \int \mathcal{D}\xi e^{-\frac{1}{2\epsilon}\int \xi^a \xi_a d\mu + \int \xi^a J_a d\mu}\delta(\mathrm{Pexp}\oint_\gamma A, g_1). \tag{15}$$

We have

$$\delta(h, g_1) = \sum_{\lambda\in\hat{G}} \chi_\lambda(h)\chi_\lambda(g_1^{-1}) \tag{16}$$

where the summation is over all irreducible unitary representation of the group , and χ_λ is the character of the representation. We then use the fermionic path integral representation of the Wilson loop [7, 12]

$$\chi_\lambda(Pexp\oint_\gamma A) = \int \mathcal{D}\eta\mathcal{D}\bar{\eta}e^{\int_0^1 dt\bar{\eta}(t)\dot{\eta}(t)+\oint_\gamma \bar{\eta}(t)A(t)\eta(t)}\bar{\eta}(1)\eta(0), \tag{17}$$

where η is a Grassmann valued vector in the representation λ.We also use the Schwinger-Fock gauge:

$$A_\mu^a(x) = \int_0^1 dssx^\nu F_{\nu\mu}^a(sx). \tag{18}$$

In this gauge , one can write

$$\oint_\gamma \bar{\eta}(t)A(t)\eta(t) = \int_D \bar{\eta}F\eta. \tag{19}$$

Using this , it is easily seen that

$$\psi_D[J] = \sum_\lambda \chi_\lambda(g_1^{-1})\int \mathcal{D}\eta\mathcal{D}\bar{\eta}e^{\int_0^1 dt\bar{\eta}(t)\dot{\eta}(t)}e^{\frac{\epsilon}{2}\int(J^a+\bar{\eta}T_\lambda^a\eta)(J_a+\bar{\eta}T_{a\lambda}\eta)\sqrt{g}dsdt}\bar{\eta}(1)\eta(0), \tag{20}$$

where $T_{a,\lambda}$'s are the generators of the group in the representation λ. This integral is also calculated to be (see the appendix of [8])

$$\psi_D[J] = Z_1[J]\psi_{2,D}[J], \tag{21}$$

where

$$Z_1[J] = e^{\frac{\epsilon}{2}\int J^a J_a d\mu}, \tag{22}$$

and

$$\psi_{2,D}[J] = \sum_\lambda \chi_\lambda(g_1^{-1})e^{-\frac{\epsilon}{2}c_2(\lambda)A(D)}\chi_\lambda(\mathcal{P}\exp\epsilon\int dt\int ds\sqrt{g}J(s,t)). \tag{23}$$

Here the ordering is according to t (the angle coordinate). The disk is parametrized by the coordinates s (the radial coordinate) and t (the angle coordinate) , and $c_2(\lambda)$ is the quadratic Casimir of the representation λ.

It is now easy to see that

$$< \xi^{a_1}(x_1)...\xi^{a_n}(x_n) >_{2,D} = \frac{1}{Z_D[0]}\sum_\lambda \chi_\lambda(g_1^{-1})e^{-\frac{\epsilon}{2}c_2(\lambda)A(D)}\epsilon^n\chi_\lambda(T^{a_1}...T^{a_n}) \tag{24}$$

for $\quad t(x_1) < ... < t(x_n)$.

To find the correlators on an arbitrary closed surface , it suffices to glue the disk to $\Sigma_{g,1}$, a genus-g surface with a boundary , with boundary condition $\mathrm{Pexp}\oint_\gamma A = g_1^{-1}$:

$$< \xi^{a_1}(x_1)...\xi^{a_n}(x_n) >_{2,\Sigma_g} = \frac{1}{Z_{\Sigma_g}[0]}\int dg_1 < \xi^{a_1}(x_1)...\xi^{a_n}(x_n) >_{2,D} \psi_{\Sigma_{g,1}}(g_1^{-1}), \tag{25}$$

where (from [7]) we have

$$Z_{\Sigma_g}[0] = \sum_\lambda d(\lambda)^{2-2g} e^{-\frac{\epsilon}{2}c_2(\lambda)A(\Sigma_g)}, \tag{26}$$

in which $d(\lambda)$ is the dimension of the representation λ , and

$$\psi_{\Sigma_{g,1}}(g_1^{-1}) = \sum_\lambda d(\lambda)^{2-2g-1}\chi_\lambda(g_1)e^{-\frac{\epsilon}{2}c_2(\lambda)A(\Sigma_{g,1})}. \tag{27}$$

Using the orthogonality relation

$$\int \chi_\lambda(g)\chi_\mu(g^{-1})dg = \delta_{\lambda\mu}, \tag{28}$$

one finds

$$< \xi^{a_1}(x_1)...\xi^{a_n}(x_n) >_{2,\Sigma_g} = \frac{1}{Z_{\Sigma_g}[0]}\sum_\lambda d(\lambda)^{2-2g-1}e^{-\frac{\epsilon}{2}c_2(\lambda)A(\Sigma_g)}\chi_\lambda(T^{a_1}...T^{a_n}), \tag{29}$$

where it is understood that $t_1 < ... < t_n$. This result obviously depends on the choice of the coordinates and hence is not gauge invaraint.

Z_1 is the partition function of a free theory. So again we have (like eq.(8))

$$< \xi^{a_1}(x_1)...\xi^{a_n}(x_{2n}) >_{1,\Sigma_g} = \sum_p G^{a_{i_1}a_{i_2}}(x_{i_1}, x_{i_2})...G^{a_{i_{2n-1}}a_{i_{2n}}}(x_{i_{2n-1}}, x_{i_{2n}}) \tag{30}$$

where

$$G^{ab}(x, y) = \epsilon\delta^{ab}\delta(x - y). \tag{31}$$

This completes the expression of the correlators of the Yang-Mills theory on Σ_g . Again they consist of a free part and a part which is almost x-independent , that is , it depends only on the angular ordering of the coordinates.

Now the important question that arises is that which part of the above results are gauge-invariant. First notice that the wavefunction (23) is not gauge-invariant. This can be checked by noting that it is not invariant under the transformation $J(x) \to U(x)J(x)U^{-1}(x)$ (which induces the gauge transformation) , because of the character term $\chi_\lambda(\mathcal{P}\exp\epsilon \int J)$. Now try to divide the disk D to N parts and consider the wavefunction (23) for the disk $D_N = D/N$, (with area A/N). Then try to find the wavefuction of the disk D by gluing the wavefunction of the small disks. This is justified only if the wavefunction is gauge invariant , because the radial and angle variables in each disk is not the same as the other disks. However , as N tends to infinity , it is enough to calculate the wavefunction of the small disks only up to first order of (A/N) . But , up to first order , we have

$$\psi_{2,D/N}[J] = \sum_\lambda \chi_\lambda(g_1^{-1})e^{-\frac{\epsilon}{2}c_2(\lambda)A(\frac{D}{N})}d(\lambda), \tag{32}$$

which is gauge invariant. Gluing these we find

$$Z^{G.I.}_{\Sigma_g}[J]=e^{\frac{\epsilon}{2}\int J^a J_a d\mu}Z_{\Sigma_g}[0]. \tag{33}$$

Therefore the gauge invariant part of the correlators are those quoted in eq.(30) which are free.

Another method of calculating the gauge-invariant correlators is using the expectation value of Wilson loops (which are gauge-invariant) [7] :

$$<\chi_\mu(\mathrm{Pexp}\oint_\gamma A)>=\frac{1}{Z_{\Sigma_g}[0]}\sum_\lambda\sum_{\rho\in\lambda\otimes\mu}d(\lambda)d(\rho)^{1-2g}exp\{-\epsilon[c_2(\lambda)A(D)+c_2(\rho)(A(\Sigma_g)-A(D))]\} \tag{34}$$

where $\gamma = \partial D$. If , by symmetry consideration , we use the following ansatz for the gauge-invariant two point function :

$$<\xi^a(x)\xi^b(y)>^{G.I.}=M\delta^{ab}\delta(x-y), \tag{35}$$

and , for small $A(D)$, equate the linear term of both sides of eq.(34) , we will find :

$$M=-\frac{\epsilon}{d(\mu)c_2(\mu)Z_{\Sigma_g}[0]}\sum_\lambda\sum_{\rho\in\lambda\otimes\mu}d(\lambda)d(\rho)^{1-2g}e^{-\frac{\epsilon}{2}c_2(\rho)A(\Sigma_g)}(c_2(\rho)-c_2(\lambda)). \tag{36}$$

Then , using the identities :

$$\sum_{\rho\in\lambda\otimes\mu}d(\rho)c_2(\rho)=d(\lambda)d(\mu)(c_2(\lambda)+c_2(\mu))$$

$$\sum_{\rho\in\lambda\otimes\mu}d(\rho)=d(\lambda)d(\mu), \tag{37}$$

it is seen that $M=\epsilon$, which is consistent with (31).

Acknowledgement

We would like to thank the research vice-chancellor of the university of Tehran.

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